Simulating continuous quantum systems by mean field fluctuations

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In this paper we are discussing the question how a continuous quantum system can be simulated by mean field fluctuations of a finite number of qubits. On the kinematical side this leads to a convergence result which states that appropriately chosen fluctuation operators converge in a certain weak sense (i.e. we are comparing expectation values) to canonical position and momentum Q, P of one-degree of freedom, continuous quantum system. This result is substantially stronger than existing methods which rely either on central limit theorem arguments (and are therefore restricted to the Gaussian world) or are valid only if the states of the ensembles are close to the "fully polarized" state. Dynamically this relationship keeps perfectly intact (at least for small times) as long as the continuous system evolves according to a quadratic Hamiltonian. In other words we can approximate the corresponding (Heisenberg picture) time evolution of the canonical operators Q, P up to arbitrary accuracy by the appropriately chosen time evolution of fluctuation operators of the finite systems.

1. Introduction

One of the most exciting, recent experiments in quantum information science is the realization of quantum memory. In [1] it is shown that the state of a laser field can be stored in collective degrees of freedom of an ensemble of about 10^{23} earth-alcaline atoms at room temperature. When the light is released after a few microseconds it is in a state which has a very good fidelity with respect to the original state.

A crucial part of the theoretical description of these experiments is the Holstein-Primakoff transformation [2], which basically implies that under certain conditions the ensemble can be treated like a *continuous* quantum system interacting harmonically with the light field. Storing

and releasing of the light is then – roughly speaking – just a swap of two coupled oscillators; cf [3] for details.

One of the most important assumptions we have to make for this approximation to be valid is invariance of the interaction under permutations of the atoms (i.e. the intensity of the laser field is constant over the region where the ensemble is spread), and this brings the whole model in contact with mean field theory and mean field fluctuations. The Holstein-Primakoff transformation can be regarded in this context as a special instance of the fact that fluctuation operators (which measure the fluctuations of a mean field variable around its expectation value) satisfy in the infinite particle limit and under quite general conditions canonical commutation relations [4].

This observation provides a big motivation to revisit the subject of mean field fluctuations and to look how it can be used to describe the type of matter-light interactions used in quantum memory experiments. Most of the results available in the literature are mainly targeted towards the needs of statistical mechanics [5] and are strongly connected to the realm of the central limit theorem [6]. This implies in particular that in the infinite particle limit we only get Gaussian states. For the type of applications we have in mind this is a too severe restriction. To remove it is one main goal of this paper.

To this end we are going to ask how a continuous quantum system can be simulated by mean field fluctuations of a finite number of qubits. This question involves kinematical as well as dynamical aspects. On the kinematical side this means in particular to give the above statement about commutation relations of fluctuation operators a more detailed and mathematical precise meaning. This is done by a convergence result (Theorem 2.9) which states (roughly speaking) that appropriately chosen fluctuation operators converge in a certain weak sense (i.e. we are comparing expectation values) to canonical position and momentum Q, P of one-degree of freedom, continuous quantum system. This result is substantially stronger than existing methods which rely either on central limit theorem arguments (and are therefore restricted to the Gaussian world) or are valid only if the states of the ensembles are close to the "fully polarized" state. As a side result we will see that the mixedness of the one particle states (i.e. the restriction of the global state to a single particle) is related to an effective $\hbar$ in the corresponding continuous system.

Dynamically this relationship keeps perfectly intact (at least for small times) as long as the continuous system evolves according to a quadratic Hamiltonian. In other words we can approximate the corresponding (Heisenberg picture) time evolution of the canonical operators Q, P up to arbitrary accuracy by the appropriately chosen time evolution of fluctuation operators of the finite systems (Theorem 2.10). This is, again, a big improvement over existing results where only special Hamiltonians and weaker types of approximations are discussed.

Before we start to present the details, let us have a short look on the organization of the paper. Section 2 collects all main results of the paper without proofs and keeping the technical overhead as small as possible. It consists of three subsections starting with 2.1 where fluctuation operators are reviewed and slightly reformulated for the needs of this paper. Subsection 2.2 discusses Schwartz operators recently introduced in [7]. This class encompasses all states of the continuous system which can be simulated with our method. In Subsection 2.3 we present a new representation of permutation invariant states of many qubit systems and introduce a special type of convergence. It plays a crucial role in the two main theorem which are stated and discussed in Subsection 2.4. The rest of the paper is devoted to proofs (Section 3) and an outlook to further research (Section 4).

2. Basic definitions and main results

2.1. Fluctuation operators

Let us start to setup some notations and to recall a few known results which will be important for the rest of this paper. Therefore lets denote the space of (bounded) operators on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ by $\mathcal{B}(\mathcal{H})$. For each $A \in \mathcal{B}(\mathcal{H})$ and a fixed density operator θ (the *reference state*) on $\mathcal{H}$ define the *fluctuation operator* $F_M(A) \in \mathcal{B}(\mathcal{H}^{\otimes M})$ by

$$F_M(A) = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} \Big(A^{(i)} - \operatorname{Tr}(A\theta)\mathbb{1} \Big) \tag{1}$$

where

$$A^{(i)} = \mathbb{1}^{\otimes(i-1)} \otimes A \otimes \mathbb{1}^{M-i}. \tag{2}$$

The $F_M(A)$ are important in mean field theory and describe the quantum fluctuations around a mean field observables; cf [5] for details. For us the $F_M(A)$ will serve as approximations (in a sense we will make clear below) of canonical operators of a continuous quantum system. To this end the central definition is the following:

Definition 2.1. *A sequence of density operators ρ_M, $M \in \mathbb{N}$ has $\sqrt{M}$-*fluctuations *if*

$$\lim_{M\to\infty} \operatorname{Tr}\big(\rho_M F_M(A_1) \cdots F_M(A_K)\big) \tag{3}$$

exists and is finite for all $K \in \mathbb{N}$ and all $A_1, \dots, A_K \in \mathcal{B}(\mathcal{H})$.

For the rest of this paper let us choose a reference state θ satisfying (where σ_α, $\alpha = 1, 2, 3$ denote Pauli matrices)

$$\operatorname{Tr}(\theta\sigma_1) = \operatorname{Tr}(\theta\sigma_2) = 0, \quad \operatorname{Tr}(\theta\sigma_3) = 2\lambda, \ \lambda \in [0, 1] \tag{4}$$

and consider fluctuation operators corresponding to σ_1 and σ_2.

$$Q_M = F_M\left(\frac{\sigma_1}{\sqrt{2}}\right) = \frac{L_{M,1}}{\sqrt{2M}} \quad P_M = F_M\left(\frac{\sigma_2}{\sqrt{2}}\right) = \frac{L_{M,2}}{\sqrt{2M}} \tag{5}$$

here $L_{M,\alpha}$ denote global angular momentum (or global spin) operators given by

$$L_{\alpha,M} = \frac{1}{2} \sum_i \sigma_\alpha^{(i)}, \quad \alpha = 1, 2, 3. \tag{6}$$

The operators Q_M and P_M satisfy the commutation relations

$$[Q_M, P_M] - i\lambda\mathbb{1} = \frac{i}{2\sqrt{M}} F_N(\sigma_3). \tag{7}$$

Taking expectation values and the limit $M \to \infty$ on both sides leads to

$$\lim_{M\to\infty} \operatorname{Tr}\big(\rho_M [Q_M, P_M]\big) - i\lambda = \frac{i}{2\sqrt{M}} \lim_{M\to\infty} \operatorname{Tr}\big(\rho_M F_M(\sigma_3)\big) = 0. \tag{8}$$

This indicates that Equation (7) converges in a certain formal way towards the canonical commutation relations. A more precise version of this statement is the following (the proof is postponed to the appendix):

Proposition 2.2. *Consider a sequence $\rho_M \in \mathcal{B}(\mathcal{H}^{\otimes M})$, $M \in \mathbb{N}$ of density operators with $\sqrt{M}$ fluctuations. Then there is a Hilbert space $\mathcal{H}_\infty$, a density operator ρ_∞ and two symmetric operators Q_∞, P_∞ with the common, invariant, dense domain D such that for any polynomial $f(q,p)$ in two non-commuting variables q,p the following statements hold:*

1. $\rho_\infty D \subset D$,
2. *The closures of $f(Q_\infty, P_\infty)\rho_\infty$ and $\rho_\infty f(Q_\infty, P_\infty)$ are of trace class.*
3. $\lim_{M\to\infty} \mathrm{Tr}\big(f(Q_M, P_M)\rho_M\big) = \mathrm{Tr}\big(f(Q_\infty, P_\infty)\rho_\infty\big) = \mathrm{Tr}\big(\rho_\infty f(Q_\infty, P_\infty)\big)$,
4. $[Q_\infty, P_\infty]\phi = i\lambda\phi \ \forall \phi \in D$.

Condition 1 and 2 are of technical nature and needed to guarantee that 3 is a mathematically well defined expression. We come back to this point in the next subsection. Item 3 itself establishes the operators Q_∞ and P_∞ as a form of weak limit of the fluctuators Q_M, P_M, and condition 4 states that these limit operators satisfy the canonical commutation relations with λ from Equation (4) *as an effective $\hbar$*. We are tempted to interpret this result by saying that "the collective fluctuations of an ensemble of two-level atoms behave in the infinite particle limit like a continuous quantum system with one degree of freedom." Unfortunately this interpretation is premature, since condition 4 is not sufficient to conclude that Q_∞ and P_∞ are canonical position and momentum. It is not even guaranteed that Q_∞ and P_∞ are essentially self-adjoint on the domain D. Hence the crucial question is:

Question 2.3. *For which sequences $\rho_M \in \mathcal{B}(\mathcal{H}^{\otimes M})$, $M \in \mathbb{N}$ Proposition 2.2 holds with*

1. $\mathcal{H}_\infty = \mathrm{L}^2(\mathbb{R})$ – *i.e. the Hilbert space of square integrable functions on $\mathbb{R}$,*
2. $D = \mathcal{S}(\mathbb{R})$ – *i.e. the space of Schwartz functions on $\mathbb{R}$,*
3. $Q_\infty = Q$ *with* $(Q\phi)(x) = x\phi(x) \ \forall \phi \in \mathcal{S}(\mathbb{R})$ – *i.e. Q is the canonical position operator with $\mathcal{S}(\mathbb{R})$ as its domain.*
4. $P_\infty = \lambda P$ *with* $P\phi = -i\phi' \ \forall \phi \in \mathcal{S}(\mathbb{R})$ – *i.e. P is the canonical momentum operator with $\mathcal{S}(\mathbb{R})$ as its domain.*

One way to answer this question is to be more restrictive with the selection of the sequences of states ρ_M, $M \in \mathbb{N}$. An important case arises if take a translationally invariant state ω of an infinite spin-chain and define the ρ_M in terms of the restrictions of ω to a sub chain of length M. It is shown [6] that in the case where ω is exponentially clustering (i.e. correlations decay exponentially fast), Proposition 2.2 holds with Q_∞ and P_∞ being the canonical position and momentum operators and ρ_∞ being a Gaussian state.

In this paper, we want to discuss a slightly different approach which puts more emphasis on the non-Gaussian world. In the following two sections we will present in particular a different form of convergence which is tighter than pointwise convergence (as used in Definition 2.1), and which can therefore guarantee that the $M \to \infty$ limit always leads to the Schrödinger representation of the CCR.

Before we come to this point let us add a short remark about the role of the reference state θ. To this end let us calculate first the expectation value of one fluctuation operator $F_M(A)$. For an arbitrary ρ_M we have

$$\mathrm{Tr}(A^{(i)}\rho_M) = \mathrm{Tr}(A\theta_{M,i}) \tag{9}$$

where $\theta_{M,i}$ is the restriction of ρ_M to the i^{th} site (for a permutation invariant ρ_M all $\theta_{M,i}$ coincide). Hence

$$\operatorname{Tr}(F_M(A)\rho_M) = \sqrt{M}\operatorname{Tr}\big((\theta_M - \theta)A\big), \quad \theta_M = \frac{1}{M}\sum_{i=1}^{M}\theta_{M,i}. \tag{10}$$

Hence, $\lim_{M\to\infty}\operatorname{Tr}(F_M(A)\rho_M) < \infty$ for all $A \in \mathcal{B}(\mathcal{H})$ can hold only if

$$\lim_{M\to\infty}\theta_M = \theta \tag{11}$$

is satisfied. In other words: if a sequence ρ_M has $\sqrt{M}$ fluctuations, its averaged one-site restriction in the limit $M \to \infty$ is given by θ. With the choice from Equation (4) this implies that the parameter λ – which plays the role of an *effective* $\hbar$ in Condition 4 of Proposition 2.2 – is a property of *the sequence*, while the operators Q_M, P_M do not depend on λ.

2.2. Schwartz operators

Conditions 1 and 2 of Proposition 2.2 are needed to guarantee that the traces on the right hand site of the expression in item 3 are well defined. They are, however, quite restrictive and lead to a new class of operators which are called Schwartz operators.

Definition 2.4. *An operator $\rho \in \mathcal{B}(\mathcal{H}_\infty)$ is called a* Schwartz operator *if $P^\alpha Q^\beta \rho Q^{\beta'} P^{\alpha'}$ is for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}_0$ a trace-class operator. The set of all Schwartz operators will be denoted by $\mathcal{S}(\mathcal{H}_\infty)$.*

Schwartz operators are introduced and discussed in [7]. For our purpose only few properties are needed which we will present in the following. For more details (in particular and the many degrees of freedom version) we will refer the reader to [7]. Most important for us is the fact that $\mathcal{S}(\mathcal{H}_\infty)$ admits a natural topology.

Proposition 2.5. *For each $\alpha, \alpha', \beta, \beta' \in \mathbb{N}_0$ the functional*

$$\mathcal{S}(\mathcal{H}_\infty) \ni \rho \mapsto \| \rho \|_{\alpha\alpha'\beta\beta',1} = \| P^\alpha Q^\beta \rho Q^{\beta'} P^{\alpha'} \|_1 \in \mathbb{R}^+ \tag{12}$$

is a seminorm. $\mathcal{S}(\mathcal{H}_\infty)$ together with this family is a Fréchet space.

In passing let us add the remark that we could replace the trace norm in (12) by any other p-norm (with $p > 1$ including the operator norm) and we would still get valid families of seminorms (including in particular the property that all the seminorms are finite for all Schwartz operators) defining the same topology as the family we have chosen; cf [7] for details.

There are a number of alternative characterizations of Schwartz operators and some of them we will encounter later in this paper. For now we only need a statement about some matrix elements. To this end let us introduce some notation first. The usual creation and annihilation operators are

$$a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) \tag{13}$$

both with $\mathcal{S}(\mathbb{R})$ as their domain. They give rise to the number operator[1]

$$\mathbf{N} = \overline{a^\dagger a}, \quad \mathbf{N}\psi_n = n\psi_n, \quad n \in \mathbb{N}_0 \tag{14}$$

and its eigenfunction ψ_n, $n \in \mathbb{N}_0$, i.e. ψ_n denotes the n^{th} order *Hermite function*. Now we have the following proposition [7].

[1] The overline means – as usual – the closure.

Proposition 2.6. *A trace class operator ρ is a Schwartz operator iff*

$$\sup_{n,m\in\mathbb{N}_0} |\langle\psi_n,\rho\psi_m\rangle|\cdot(|n|+|m|)^k < \infty \quad \forall k\in\mathbb{N}_0. \tag{15}$$

This implies in particular that all operators ρ where only a finite number of matrix elements $\langle\psi_n,\rho\psi_m\rangle$ are different from 0, are Schwartz operators.

2.3. Permutation invariant states

Let us come back now to Question 2.3. The purpose of this section is to define a notion of convergence which guarantees that the operators Q_∞ and P_∞ in Proposition 2.2 can be chosen as canonical position and momentum Q and λP. To this end let us have a another look at Definition 2.1. The crucial objects in Equation (3) are expectation values of products of fluctuation operators $F_M(A_M)$. The $F_M(A)$, however, commute with all permutation operators. Therefore the expectation values of the form $\mathrm{Tr}(F_M(A_1)\cdots F_M(A_K)\rho_M)$ do not change, if we replace the density operator $\rho_M\in\mathcal{B}_*(\mathcal{H}^{\otimes M})$ with its average over all permutations. In other words: As long as we are only interested in expectation values we can assume without loss of generality that ρ_M is permutation invariant.

The general structure of permutation invariant density matrices can be easily deduced from representation theory of SU(2) and the permutation group (this is very well known stuff; cf. [8] for a review and further references). The M-fold Hilbert space $\mathcal{H}^{\otimes M}$ can be decomposed such that for any $U\in\mathrm{SU}(2)$ we have

$$\mathcal{H}^{\otimes M}=\bigoplus_j \mathcal{H}_j\otimes\mathcal{K}_{M,j},\quad U^{\otimes M}=\pi_j(U)\otimes\mathbb{1} \tag{16}$$

where $\mathcal{H}_j$ is a $2j+1$ dimensional Hilbert space, $\pi_j:\mathrm{SU}(2)\to\mathcal{B}(\mathcal{H}_j)$ denotes the irreducible spin-j representation, $\mathcal{K}_{M,j}$ is a multiplicity space carrying an irreducible representation of the permutation group S_M, and the index j runs over the integers $0,1,\dots M/2$ (if M is even) or the half-integers $1/2,3/2,\dots,M/2$ (if M is odd). A permutation invariant density matrix has the form

$$\rho_M=\bigoplus_j w_{M,j}\left(\rho_{M,j}\otimes\frac{\mathbb{1}}{\dim\mathcal{K}_{M,j}}\right) \tag{17}$$

where $\rho_{M,j}$ is a density matrix on $\mathcal{H}_j$ and the weights $w_{M,j}$ are positive real numbers satisfying

$$\sum_j w_{M,j}=1. \tag{18}$$

If $w_{M,j}=0$ for some j we set $\rho_{M,j}=0$, too.

In a similar way we can decompose the angular momentum operators $L_{\alpha,N}$. If we write[2] $L_\alpha^{(j)}=\pi_j(\sigma_\alpha/2)\in\mathcal{B}(\mathcal{H}_j)$ for angular momentum in the spin-j representation we get

$$L_{\alpha,N}=\bigoplus_j L_\alpha^{(j)}\otimes\mathbb{1}. \tag{19}$$

Of special importance as well are the eigenvectors $\psi_n^{(j)}$ of $L_3^{(j)}$

$$L_3^{(j)}\psi_n^{(j)}=(j-n)\psi_n^{(j)},\quad n=0,\dots,2j \tag{20}$$

[2]In abuse of notation we use the symbol for the representation of the group and the corresponding Lie-algebra.

and the ladder operators $L_{\pm}^{(j)} = L_1^{(j)} \pm i L_2^{(j)}$

$$L_+^{(j)} \psi_n^{(j)} = \sqrt{n(2j-n+1)}\psi_{n-1}^{(j)} \quad L_-^{(j)} \psi_n^{(j)} = \sqrt{(2j-n)(n+1)}\psi_{n+1}^{(j)}. \tag{21}$$

Th next important points concerns the relation between the Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H}_\infty$. The former is the representation space of the spin-j irreducible SU(2) representation and contains the distinguished basis $\psi_n^{(j)}$. The latter was introduced in Subsection 2.2 as $\mathrm{L}^2(\mathbb{R})$ and we chose the *Hermite functions* ψ_n, $n \in \mathbb{N}$ as a basis (cf. Equation (14)). Now we embed all the $\mathcal{H}_j$ into $\mathcal{H}_\infty$ such that $\psi_n^{(j)}$ becomes for all $n = 0, \dots, 2j$ the Hermite function ψ_n. In other words, we consider the embedding:

$$\mathcal{H}_j \ni \sum_{n=0}^{2j} \phi^n \psi_n^{(j)} \mapsto \sum_{n=0}^{2j} \phi^n \psi_n \in \mathcal{H}_\infty. \tag{22}$$

In the following we will *identify $\mathcal{H}_j$ with its image under this isometry.* In this way all operators ρ_M become finite rank operators on the same infinite dimensional Hilbert space $\mathcal{H}_\infty$. The same is true for the angular momentum operators $L_\alpha^{(j)}$, $\alpha = 1, 2, 3$ and $L_\pm^{(j)}$.

Another crucial step is a rescaling of the parameter $j = 0, \dots, M/2$ (or $1/2, \dots, M/2$) by

$$x_j = \frac{2j}{M} \quad j_x = \frac{\lfloor Mx \rfloor}{2}. \tag{23}$$

Hence $\rho_{M,j}$ becomes ρ_{M,x_j} and if we extend it continuously to the interval $[0,1]$ we get a continuous function

$$R_M : [0,1] \ni x \mapsto R_M(x) \in \mathcal{S}(\mathcal{H}_\infty), \quad R_M(x_j) = \rho_{M,j}. \tag{24}$$

Note that R_M is not uniquely defined by ρ_M, however we can choose such a function for any ρ_M (e.g. by linear interpolation). Also note that the definition of these functions requires the embedding of the $\mathcal{H}_j$ into $\mathcal{H}_\infty$ as described above.

The advantage of this formulation just introduced is that sums over j can now be reformulated as integrals with respect to the probability measure μ_M given by

$$\int_0^1 f(x)\mu_M(dx) = \sum_j w_{M,j} f(x_j). \tag{25}$$

To see this consider the expectation value of a permutation invariant observable (like angular momentum introduced above) $A_M = \sum A_{M,j} \otimes \mathbb{1}$ in the state ρ_M. Using the decomposition in Equation (17) we get

$$\mathrm{Tr}(\rho_M A_M) = \sum_j w_{M,j} \, \mathrm{Tr}(\rho_{M,j} A_{M,j}) = \int_0^1 \mathrm{Tr}(R_M(x) A_M(x)) \mu_M(dx) \tag{26}$$

where we have used again a continuous function $A_M : [0,1] \to \mathcal{B}(\mathcal{H}_\infty)$ satisfying $A_M(x_j) = A_{M,j}$. This motivates the following definition:

Definition 2.7. *Given ρ_M, permutation invariant. We call a continuous function $R_M : [0,1] \to \mathcal{S}(\mathcal{H}_\infty)$ an* ***integral representation*** *of ρ_M, if $R_M(2j/M) = \rho_{M,j}$, where $\rho_{M,j} \in \mathcal{B}_*(\mathcal{H}_j)$ are the density operators in the decomposition from Equation (17).*

Note again that an integral representation always exist, but is never unique. The latter is not a problem, though, since integrals as in Equation (26) only depend on ρ_M and not on the integral representation chosen.

Finally let us have a look at the convergence of sequences of permutation invariant density operators ρ_M, $M \in \mathbb{N}$. We can split this question up into two pieces: convergence of the sequence of measures $\mu_{\rho,M}$ and convergence of the sequence of functions R_M. A particular behavior is described by the next definition, which captures the situation that in the limit $M \to \infty$ only one value $x = \lambda \in [0,1]$ is relevant, while all other x does not matter. Or to state it in a different way: For large M the state ρ_M describes a very sharp angular momentum $\vec{L}^2$ with j centered around $j_\lambda = \lambda M/2$. For these j the corresponding $\rho_{M,j}$ is close to ρ_∞ (in the topology of $\mathcal{S}(\mathcal{H}_\infty)$ which measures in turn the moments). All other ρ_j are unimportant.

Definition 2.8. *We say that a sequence $\rho_M \in \mathcal{B}_*(\mathcal{H}^{\otimes M})$ of permutation invariant density operators* converges at $\lambda \in [0,1]$ towards *a state $\rho_\infty \in \mathcal{S}(\mathcal{H}_\infty)$ if each ρ_M admits an integral representation R_M such that the following conditions hold:*

1. *The sequence of probability measures μ_M converges weakly to the point measure to the point measure at λ that is*
$$\lim_{M\to\infty} \int_0^1 f(x)\mu_M(dx) = f(\lambda) \tag{27}$$
for all continuous functions $f : [0,1] \to \mathbb{R}$

2. *The set*
$$\{R_M(x) | M \in \mathbb{N}, x \in [0,1]\} \subset \mathcal{S}(\mathcal{H}_\infty) \tag{28}$$
is bounded.

3. *There is a neighbourhood of $I \subset [0,1]$ of λ and a continuous function $R_\infty : I \to \mathcal{S}(\mathcal{H}_\infty)$ such that $R_\infty(\lambda) = \rho_\infty$ and*
$$\lim_{M\to\infty} R_M(x) = R_\infty(x) \textit{ uniformly on } I \tag{29}$$
holds in the topology of $\mathcal{S}(\mathcal{H}_\infty)$.

2.4. The main result

Now we can come back to Question 2.3. The next theorem simply says that the convergence given in Definition 2.8 provides (at least) a sufficient condition (all proofs are postponed to Section 3).

Theorem 2.9. *For any sequence $\rho_M \in \mathcal{B}_*(\mathcal{H}^{\otimes M})$, $M \in \mathbb{N}$ of permutation invariant states converging towards a $\rho_\infty \in \mathcal{S}(\mathcal{H}_\infty)$ at $\lambda \in (0,1]$, and all polynomials $f(q,p)$ in two non-commuting variables we have*
$$\lim_{M\to\infty} \mathrm{Tr}(\rho_M f(Q_M, P_M)) = \mathrm{Tr}(\rho_\infty f(Q_\infty, P_\infty)). \tag{30}$$
with $Q_\infty = \sqrt{\lambda} Q$ and $P_\infty = \sqrt{\lambda} P$.

Note that convergence at λ in the sense of Definition 2.8 *does not imply* $\sqrt{M}$ fluctuations. The reason is that we are not controlling convergence of the moments of the measures μ_M. Therefore expectation values of products involving $F_M(\sigma_3)$ can still diverge. It is easy to fix this by considering a stronger type of convergence for the μ_M. However, this is not needed for getting the proper limit of the Q_M, P_M.

One way to interpret Theorem 2.9 is in terms of *quantum simulation*: If we are interested in the expected value of an observable $X = f(Q_\infty, P_\infty)$ of a continuous quantum system in a state ρ_∞, we can measure instead the observable $X_M = f(Q_M, P_M)$ of an ensemble of M qubits in the state ρ_M and the error we make that way can be arbitrarily small, provided M is big enough.

It is possible to simulate each density operator $\rho_\infty \in \mathcal{S}(\mathcal{H}_\infty)$ that way. To see this consider for each half-integer j the projection

$$E_j : \mathcal{H}_\infty \to \mathrm{span}\{\psi_n \,|\, 0 \leq n \leq 2j\} \tag{31}$$

and a sequence $j_M \in \mathbb{R}$ satisfying $j_M \in \{0, \dots, M/2\}$ if M is even, $j_M \in \{1/2, \dots, M/2\}$ if M is odd, and $\lim_{M\to\infty} 2j_M/M = \lambda$. Now we can define

$$\tilde{\rho}_M = E_{j_M}\rho_\infty E_{j_M} \otimes \mathbb{1} \in \mathcal{B}(\mathcal{H}_{j_M} \otimes \mathcal{K}_{M,j_M}) \subset \mathcal{B}(\mathcal{H}^{\otimes M}) \tag{32}$$

and

$$\rho_M = \frac{\tilde{\rho}_M}{\mathrm{Tr}(\tilde{\rho}_M)} \quad \text{if } \mathrm{Tr}(\tilde{\rho}_M) \neq 0, \tag{33}$$

where we have used the identification of SU(2) representation space $\mathcal{H}_{j_M}$ with the subspace $E_{j_M}\mathcal{H}_\infty$ introduced in Subsection 2.3. If $\mathrm{Tr}(\tilde{\rho}_M) = 0$ holds (this can only happen for finitely many M) we can just choose an arbitrary density operator. It is obvious that the sequence just constructed converges towards ρ_∞ at λ and therefore it simulates ρ_∞ in the sense described in the last paragraph.

The parameter λ can be freely chosen in this construction (by choosing the sequence j_M) without affecting the simulability of ρ_∞. This seems to indicate that λ does not play an important role, but this impression is wrong. At the end of Subsection 2.1 we have seen that λ describes (asymptotically, i.e. in the limit $M \to \infty$) the noise in the one-site restrictions of the ρ_M, ranging from "fully polarized" for $\lambda = 1$ and "fully depolarized" for $\lambda = 0$.

At the same time it plays in the commutation relations in Proposition 2.2 the role of an effective $\hbar$, and this also holds for the operators Q_∞ and P_∞ in Theorem 2.9. However, in contrast to our expectation in Question 2.3 Q_∞ and P_∞ are not exactly canonical position and momentum (except in the case $\lambda = 1$), they differ by a factor $\lambda^{-1/2}$ and $\lambda^{1/2}$ respectively. Hence, although λ has *some* properties of $\hbar$ the limit $\lambda \to \infty$ is not the classical limit (unless we rescale the finite dimensional observables Q_M and P_M appropriately). Instead all observables vanish if $\lambda \to \infty$. Nevertheless, the case $\lambda = 0$ is interesting (in particular because it is important experimentally [9]) and therefore deserves a more detailed study in a forthcoming paper. We come back to this point in Section 4.

Now let us come back to the simulation point of view. We have argued that we can simulate states, but what about time-evolutions? To answer this question, consider a self-adjoint[3] polynomial $h(q,p)$ and define the Hamiltonians $H_M = h(Q_M, P_M)$ and $H = h(Q, P)$. Theorem 2.9 shows that we can simulate expectation values of H by expectation values of H_M. But what about time-evolutions? This question is much more difficult. The first big obstacle arises from the fact that H (with $\mathcal{S}(\mathbb{R})$ as its domain) is only symmetric but in general not essentially selfadjoint. It is even possible that no self-adjoint extensions exist at all [10]. In these cases the unitaries

$$U_{M,t} = \exp(-itH_M) \tag{34}$$

(which always exist, since the underlying Hilbert space is finite dimensional) do approximate a reasonable time-evolution of the continuous system. However, what about the case where H is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$? In this case we can define the weakly continuous one-parameter group

$$U_{\lambda,t} = \exp(-it\lambda H) \tag{35}$$

[3]For each polynomial $f(q,p)$ in two non-commutative variable we can construct the adjoint we apply complex conjugation to the coefficients and reverse the ordering of each monomial. A polynomial is self-adjoint, if it coincides with its adjoint.

and ask whether the finite dimensional dynamics $U_{M,t}$ approximates the $U_{\lambda,t}$ with a certain error for a finite amount of time. The next theorem shows that this indeed the case for quadratic Hamiltonians.

Theorem 2.10. *Consider: A sequence $\rho_M \in \mathcal{B}_*(\mathcal{H}^{\otimes M})$, $M \in \mathbb{N}$ of permutation invariant states converging towards a $\rho_\infty \in \mathcal{S}(\mathcal{H}_\infty)$ at $\lambda \in (0,1]$, a second order, self adjoint, homogeneous polynomial $h(q,p)$ and the corresponding unitaries, as defined in Equations (34) and (35). Then there is $t_0 > 0$ such that for all $t \in \mathbb{R}$ with $|t| < t_0$ and all polynomials $f(q,p)$ we have*

$$\lim_{M\to\infty} \mathrm{Tr}(U_{M,t}\rho_M U_{M,t}^* f(Q_M, P_M)) = \mathrm{Tr}(U_{\lambda,t}\rho_\infty U_{\lambda,t}^* f(Q_\infty, P_\infty)). \tag{36}$$

with $Q_\infty = \sqrt{\lambda}Q$ and $P_\infty = \sqrt{\lambda}P$.

This theorem implies that the time evolution of the expectation values $\langle f(Q,P)\rangle$ of any observable $f(Q,P)$ can be approximated for short times and with a bounded error ϵ by the time evolution of the corresponding expectation value $\langle f(Q_M, P_M)\rangle$ of the finite dimensional system. The size of the error can be as small as possible provided the number M of qubits is big enough. This strengthens the simulation point of view introduced above (at least as long as quadratic Hamiltonians are considered).

There is, however, one small problem with this result, and this is the time limit t_0. As long as we are considering an ensemble of *fixed* size M it is natural that we can keep a bounded error ϵ only for a finite amount of time. The bound t_0 in theorem, however, holds also *in the infinite particle limit*. This is counter intuitive since we would assume that we can always improve the quality the simulation by increasing the number M of particles. Therefore, it is likely that the restriction of a finite t_0 is only a restriction of the proof and not of the result.

3. Fluctuator dynamics

The purpose of this section is to prove Theorem 2.10 and in this context we will learn more details about the dynamics of fluctuation operators. As another side effect, we will get a proof of Theorem 2.9 as well, since it arises from 2.10 in the special case $t = 0$. Let us start with some additional notations.

- We need the functions

$$\beta : \mathbb{N}^2 \times [0,1] \to [0,1] : (M,n,x) \mapsto \beta_M(x,n) = \theta_1(x - \frac{n}{M\lambda}) \quad \text{with } \theta_1(x) = \sqrt{x\chi_{[0,1]}(x)}, \tag{37}$$

  where χ_S is the characteristic function of the set S. Some of the arguments of β will be often omitted when confusion can be avoided.

- Furthermore we introduce $a_M : [0,1] \to \mathcal{S}(\mathcal{H}_\infty), \forall M \in \mathbb{N}$ by

$$a_M(x) = \beta_M(x, \mathbf{N})a = a\beta_M(x, \mathbf{N} - \mathbb{1}) \tag{38}$$

$$a_M^*(x) = \beta_{M,\lambda}(x, \mathbf{N} - \mathbb{1})a^* = a^*\beta_M(x, \mathbf{N}) \tag{39}$$

  where a, a^* are the standard annihilation and creation operators, respectively, and $\mathbf{N} = a^*a$ is the number operator (cf. Section 2.2) and β with an operator argument is understood in the sense of functional calculus.

- We will also need the integrated version of $a_M(x)$, $a^*_M(x)$, which are given by

$$a_M = \frac{L_{M,+}}{\sqrt{M}} = \frac{1}{\sqrt{2}}(Q_M + iP_M), \quad a^*_M = \frac{L_{M,-}}{\sqrt{M}} = \frac{1}{\sqrt{2}}(Q_M - iP_M) \tag{40}$$

with

$$L_{M,\pm} = L_{M,1} \pm iL_{M,2} = \frac{1}{2}\sum_{j=1}^{M} \sigma_\pm^{(j)} \tag{41}$$

with $\sigma_\pm = \sigma_1 \pm i\sigma_2$ in terms of Pauli matrices. a_M, a^*_M are related to the *functions* $a_M(x)$, $a^*_M(x)$ by

$$a_M = \bigoplus_j a_M(x_j) \otimes \mathbb{1}, \quad a^*_M = \bigoplus_j a^*_M(x) \otimes \mathbb{1}. \tag{42}$$

where x_j is given as in Equation (23) by $x_j = 2j/M$.

- The quadratic Hamiltonian H will be written as

$$H = \sum_{k=0}^{3} c_k A_k \text{ with } A_0 = a^2, A_1 = aa^*, A_2 = a^*a, A_3 = a^{*2} \tag{43}$$

and the finite dimensional version $\mathcal{H}_M$ becomes similarly

$$H_M = \sum_{k=0}^{3} c_k A_{M,k} \text{ with } A_{M,0} = a_M^2,\ A_{M,1} = a_M a^*_M,\ A_{M,2} = a^*_M a_M,\ A_{M,3} = a^{*2}_M. \tag{44}$$

In analogy to Equation (42) we can rewrite H_M as a direct sum

$$H_M = \bigoplus_j H_M(x_j) \otimes \mathbb{1} \tag{45}$$

where

$$H_M(x) = \sum_{j=0}^{3} c_j B_{M,j}(x) A_j = \sum_{j=0}^{3} c_j A_{M,j}(x), \tag{46}$$

with

$$\begin{aligned} B_{M,0}(x) &= \beta_M(x, \mathbf{N})\beta_M(x, \mathbf{N} + \mathbb{1}) & B_{M,1}(x) &= \beta_M(x, \mathbf{N})^2 \\ B_{M,2}(x) &= \beta_M(x, \mathbf{N} - \mathbb{1})^2 & B_{M,3}(x) &= \beta_M(x, \mathbf{N} - \mathbb{1})\beta_M(x, \mathbf{N} - 2\mathbb{1}) \end{aligned} \tag{47}$$

- Finally we can write the unitary $U_{M,t}$ from Equation (34) in exactly the same way

$$U_{M,t} = \bigoplus_j U_{M,t}(x) \otimes \mathbb{1} \tag{48}$$

with

$$U_{M,t}(x) = \exp\bigl(-itH_M(x)\bigr). \tag{49}$$

- Furthermore we define the multiindex $R \in \{-1, 1\}^d$, which enables the compact notation

$$a^R \equiv a_M^{(R_1)} a_M^{(R_2)} \dots a_M^{(R_d)} \quad a_M^{(1)} = a^*_M \quad a_M^{(-1)} = a_M \tag{50}$$

and similarly for the quantities $a^R \in \mathcal{S}(\mathcal{H}_\infty)$. In addition we define $|R| \equiv d$ and $w(R) \equiv \sum_i^d R_i$.

- To enable compact notations in some proofs we will write objects describing the continuous system sometimes with the subscript ∞, i.e.

$$a_\infty(x) = \sqrt{x}a, \quad a^*_\infty(x) = \sqrt{x}a^*, \quad H_\infty(x) = xH, \quad U_{\infty,t}(x) = U_{x,t}, \tag{51}$$

where x is as usual in the interval $[0, 1]$.

To prove Theorem 2.10 we have to show that for each $\epsilon > 0$ we can find an M_ϵ such that

$$\left|\mathrm{Tr}(U_{M,t}\rho_M U^*_{M,t}a^R_M) - \mathrm{Tr}(U_{\lambda,t}\rho_\infty U^*_{\lambda,t}a^R)\right| < \epsilon \tag{52}$$

holds for all $M > M_\epsilon$. Note that we have replaced the polynomials $f(Q_M, P_M)$ and $f(Q, P)$ from Theorem 2.10 by monomials a^R_M and a^R with an arbitrary multiindex R. The general statement easily follows by linearity. To get such an estimate note first that we can rewrite traces of the form $\mathrm{Tr}(U_{M,t}\rho_M U^*_{M,t}a^R_M)$ as

$$\mathrm{Tr}(U_{M,t}\rho_M U^*_{M,t}a^R_M) = \sum_j w_M(j)\,\mathrm{Tr}\big(U_{M,t}(x_j)R_M(x_j)U^*_{M,t}(x_j)a_M(x_j)\big) \tag{53}$$

$$= \int_0^1 \mathrm{Tr}\big(U_{M,t}(x)R_M(x)U^*_{M,t}(x)a_M(x)\big)\mu_M(dx) \tag{54}$$

where R_M is an integral representation of ρ_M and μ_M is the measure define in Equation (25). With this we can rewrite the right hand side of (52) as

$$\begin{aligned}\big|\mathrm{Tr}(U_{M,t}\rho_M U^*_{M,t}a^R_M) - \mathrm{Tr}(U_{\lambda,t}\rho_\infty U^*_{\lambda,t}a^R_\infty(\lambda))\big| <& \\ \int_0^1 \big|\mathrm{Tr}\big(U_{M,t}(x)R_M(x)U^*_{M,t}(x)(a^R_M(x) - a^R_\infty(x))\big)\big|\mu_M(dx) +& \\ \int_0^1 \big|\mathrm{Tr}\big((U_{\lambda,t}\rho_\infty U^*_{\lambda,t} - U_{M,t}(x)\rho_\infty U^*_{M,t}(x))a^R_\infty(x)\big)\big|\mu_M(dx)&\end{aligned} \tag{55}$$

Note that we have used here $a^*_\infty(\lambda)$ and $a_\infty(\lambda)$ from Equation (51). They are creation and annihilation operator belonging to Q_∞ and P_∞ from Theorem 2.9 and 2.10.

The discussion of the two integrals on the right hand side is now broken up into four steps.

- In Subsection 3.1 we will collect additional material about Schwartz operators, which is used throughout this section.
- Then we show that the sequences $U^*_{M,t}a^R_M(x)U_{M,t}\psi_n$ converge for any Hermite function to $U^*_{x,t}a^RU_{x,t}\psi_n$; cf. Subsection 3.2.
- In Section 3.3 we look at operators $(N+p)^{p/2}U^*_{M,t}a^R_M(x)U_{M,t}$ and show that $p \in \mathbb{N}$ can be chosen such that the sequence of their norms is uniformly bounded in x. This allows us to trace convergence of unbounded operators back to convergence of bounded operators; cf. Subsection 3.3.
- The two previous steps together allows us to bound the integrands of the integrals in Equation (55) and this leads to the required estimate; cf. Subsection 3.4

3.1. More on Schwartz operators

Let us collect some properties of Schwartz operators which are needed for the proof. The standard reference is [7]. The first statement says that the space $\mathcal{S}(\mathcal{H}_\infty)$ is stable under multiplication with polynomials in Q and P.

Proposition 3.1. *Consider a polynomial $f(Q,P)$ in position Q and momentum P and a Schwartz operator ρ. The products $f(Q,P)\rho$ and $\rho f(Q,P)$ are closeable operators and their closures are again Schwartz operators*[4]*. The two maps*

$$\mathcal{S}(\mathcal{H}_\infty) \ni \rho \mapsto f(Q,P)\rho \in \mathcal{S}(\mathcal{H}_\infty) \tag{56}$$
$$\mathcal{S}(\mathcal{H}_\infty) \ni \rho \mapsto \rho f(Q,P) \in \mathcal{S}(\mathcal{H}_\infty) \tag{57}$$

defined that way are continuous.

Proof. This is an easy consequence of the definition. □

Furthermore the space $\mathcal{S}(\mathcal{H}_\infty)$ is closed under multiplication. In other words [7]

Proposition 3.2. *The product of two Schwartz operators is again a Schwartz operator.*

The next result is an alternative characterization of Schwartz operators. To this end let us first introduce for any $x = (x_1, x_2) \in \mathbb{R}^2$ the Weyl operator acting on $\psi \in \mathrm{L}^2(\mathbb{R})$ by

$$(W(x_1,x_2)\psi)(x) = e^{-ix_1x_2/2}e^{ix_2x}\psi(x-x_1). \tag{58}$$

The W are unitary and for any density operators ρ they give rise to the *characteristic function*

$$\mathbb{R}^2 \ni x \mapsto \hat{\rho}(x) = \mathrm{Tr}(\rho W(x)) \in \mathbb{C} \tag{59}$$

Now we have [7]:

Proposition 3.3. *A density operator ρ is a Schwartz operator iff its characteristic function $\hat{\rho}$ is a Schwartz function.*

Now let us come back to the quadratic Hamiltonians H and $H_M(x)$ defined in Equation (43) and (46), and the corresponding time evolutions $U_{\lambda,t}$ and $U_{M,t}(x)$ from (35) and (49).

Lemma 3.4. *For each Schwartz operator $\rho \in \mathcal{S}(\mathcal{H}_\infty)$ and each multiindex $R \in \{-1,1\}^d$ the following statements hold:*

1. *$U_{\lambda,t}\rho U_{\lambda,t}^*$ and $U_{M,t}(x)\rho U_{M,t}^*(x)$ are Schwartz operators.*
2. *$U_{\lambda,t}^* a^R U_{\lambda,t}\rho$ and $U_{M,t}^*(x) a^R U_{M,t}(x)\rho$ are Schwartz operators (and in particular trace class).*
3. *We have*

$$\mathrm{Tr}\big(U_{\lambda,t}^* a^R U_{\lambda,t}\rho\big) = \mathrm{Tr}\big(a^R U_{\lambda,t}\rho U_{\lambda,t}^*\big) \tag{60}$$
$$\mathrm{Tr}\big(U_{M,t}^*(x) a^R U_{M,t}(x)\rho\big) = \mathrm{Tr}\big(a^R U_{M,t}(x)\rho U_{M,t}^*(x)\big). \tag{61}$$

[4] In slight abuse of language we will shorten in the future statements of this form as: "... is a Schwartz operator".

Proof. To prove the first statement in 1 note first that for quadratic Hamiltonians the canonical operators $Q_t = U_{t,\lambda}^* Q U_{t,\lambda}$ and $P_t = U_{t,\lambda}^* P U_{t,\lambda}$ evolve according to the corresponding classical equation of motion (this is well known or otherwise an easy exercise). This implies for the Weyl operator that $U_{t,\lambda}^* W(x) U_{t,\lambda} = W(F_t x)$ holds with a linear map F_t (the classical phase space flow). Hence we get

$$\widehat{(U_{\lambda,t}\rho U_{\lambda,t}^*)} = \widehat{\rho} \circ F_t, \tag{62}$$

which implies that $\widehat{(U_{\lambda,t}\rho U_{\lambda,t}^*)}$is a Schwartz function and therefore $U_{\lambda,t}\rho U_{\lambda,t}^*$ is a Schwartz operator by Proposition 3.3.

Now consider the unitary $U_{M,t}(x)$. Since its generator $H_M(x)$ satisfies $E_{j_x} H_M(x) E_{j_x} = H_M(x)$ with $j_x = \frac{\lfloor Mx \rfloor}{2}$ from Equation (23) and the projection E_{j_x} defined in (31), we can rewrite $U_{M,t}(x)$ as

$$U_{M,t}(x) = (\mathbb{1} - E_{j_x}) + \tilde{U}_{M,t}(x), \quad \tilde{U}_{M,t}(x) = E_{j_x} U_{M,t}(x) E_{j_x}. \tag{63}$$

Hence

$$U_{M,t}(x)\rho U_{M,t}^*(x) = \\ (\mathbb{1} - E_{j_x})\rho(\mathbb{1} - E_{j_x}) + (\mathbb{1} - E_{j_x})\rho\tilde{U}_{M,t}^*(x) + \tilde{U}_{M,t}(x)\rho(\mathbb{1} - E_{j_x}) + \tilde{U}_{M,t}\rho\tilde{U}_{M,t}^*. \tag{64}$$

Since $\rho \in \mathcal{S}(\mathcal{H}_\infty)$ by assumption, Proposition 2.6 immediately implies that $(\mathbb{1} - E_{j_x})\rho(\mathbb{1} - E_{j_x})$, $(\mathbb{1} - E_{j_x})\rho$ and $\rho(\mathbb{1} - E_{j_x})$ are all Schwartz operators. The same is true for $\tilde{U}_{M,t}$ and $\tilde{U}_{M,t}^*$ (again by Proposition 2.6). Hence by Proposition 3.2 all operators on the right hand side of (64) are Schwartz operators and therefore $U_{M,t}(x)\rho U_{M,t}^*(x)$ is a Schwartz operator as well.

To show statement 2 note first that $a^R U_{t,\lambda}\rho U_{t,\lambda}^*$ is a Schwartz operator – this follows from Proposition 3.1 and statement 1 above. Applying item 1 again we see that

$$U_{\lambda,t}^* a^R U_{\lambda,t}\rho = U_{\lambda,t}^*(a^R U_{t,\lambda}\rho U_{t,\lambda}^*)U_{t,\lambda} \tag{65}$$

is a Schwartz operator as well. Similar reasoning holds for $U_{M,t}^*(x) a^R U_{M,t}(x)\rho$.

The last statement is an easy consequence of item 2 and elementary properties of the trace. □

3.2. Hermite functions and analytic vectors for quadratic Hamiltonians

The subject of this section is convergence of sequences $a^S U_{M,t}(x)\psi_n$, $M \in \mathbb{N}$ to $a^S U_{x,t}\psi_n$. This will lead to a first partial result in Proposition 3.10 stating that the $U_{M,t}(x)$ converge strongly to $U_{x,t}$. The main technical tool will be the fact that quadratic Hamiltonians admit (finite) linear combinations of Hermite functions as analytic vectors (cf. [11, Ch. 5]). The first step is a general estimate.

Lemma 3.5. *Consider a Hermite function ψ_n and a multiindex $S \in \{-1, 1\}^{2d}$. Then the following bound holds for all $M \in \mathbb{N} \cup \{\infty\}$ and $x \in [0, 1]$.*

$$\|a^S H_M(x)\psi_n\| \leq 2^{3d+\frac{n}{2}} d! m! (32 \max |c_j|)^m \tag{66}$$

with the maximum $\max |c_j|$ *taken over the* $j = 0, \ldots, 3$.

Proof. It is sufficient to show Equation (66) for the Hamiltonian $H = H_\infty(1)$ instead of $H_M(x)$ since all bounds hold also when we substitute $a_M(x)^R$ for a^R. The former differs from the latter only by a prefactor between 0 and 1. Now consider the bound

$$[\|a^R\psi_n\| \leq 2^{\frac{n}{2}} 4^d d! \ . \tag{67}$$

$$\|a^R\psi_n\| \leq \sqrt{(n+1)(n+2)\cdots(n+2d)} \leq \sqrt{2^{n+2d}}\sqrt{(2d!)} \leq 2^{\frac{n}{2}+d}2^d d! = 2^{\frac{n}{2}}4^d d! \ , \tag{68}$$

where to see the first inequality we recall

$$a^*\psi_n = \sqrt{n+1}\psi_{n+1}, \ a\psi_n = \sqrt{n}\psi_{n-1} \ , \tag{69}$$

from which the inequality is clear as it is an equality exactly in the worst case of $a^R = (a^*)^{2d}$. For the next two inequalities we used

$$\frac{(p+q)!}{p!q!} \leq 2^{p+q} \ , \tag{70}$$

(which holds since the lhs. is a term in the binomial expansion of the rhs.) with $p = n$, $q = 2d$ and $p = q = d$, respectively.

Let us now introduce the multiindex $\mathbf{j} \in \{0,1,2,3\}^m$ to enable the compact notation

$$H^m \equiv \sum_{\mathbf{j}} c_{\mathbf{j}} A_{\mathbf{j}} \text{ with } x_{\mathbf{j}} \equiv x_{j_1}x_{j_2}\ldots x_{j_m} \tag{71}$$

Now, taking into account that

$$|c_{\mathbf{j}}| \leq |\max_{j=0,\ldots,3} c_j|^m \quad \forall \mathbf{j}; \tag{72}$$

and that $A_{\mathbf{j}} = a^R$ for each $\mathbf{j}$ with an R satisfying $|R| = 2m$, thus

$$a^S A_{\mathbf{j}} = a^{R'} \text{ with } |R'| = 2(d+m) \tag{73}$$

we can calculate

$$\begin{aligned} \|a^S H_M(x)^m \psi_n\| &\leq \sum_{\mathbf{j}} |c_{\mathbf{j}}| \|a^S A_{\mathbf{j}} \psi_n\| && (74)\\ &\leq 4^m (\max|c_j|)^m 2^{\frac{n}{2}} 4^{d+m}(d+m)! && (75)\\ &\leq (32\max(|c_j|))^m 2^{3d+\frac{n}{2}} m!d! \ , && (76) \end{aligned}$$

where in the second inequality we inserted the bound (67) for $|R'| = 2(m+d)$. The third inequality is due to the application of (70) with $p = m, q = d$. □

Lemma 3.6. *The bound in (66) also holds if* $|s| = 2d-1$

Proof. We obviously have

$$\|a\,a^S H_M(x)^m\psi_n\| \geq \|a^S \tilde{H}^m \psi_n\| \ , \tag{77}$$

so the statement follows from Lemma 3.5. □

Lemma 3.7. *For each* $M, n \in \mathbb{N}$ *and* $x \in [0,1]$ *the following bound holds*

$$|\sqrt{x} - \beta_M(x,n)| \leq \sqrt{\frac{n}{M}} \tag{78}$$

Proof. Recall the definition of β (cf. Equation (37):

$$\beta : \mathbb{N}^2 \times [0,1] \to [0,1] : (M,n,x) \mapsto \beta_M(x,n) = \theta_1(x - \frac{n}{M}) \text{ with } \theta_1(x) = \sqrt{x\chi_{[0,1]}(x)}. \tag{79}$$

If $x < n/M$ the argument of the root is negative and we get $\beta_M(x,n) = 0$, hence

$$|\sqrt{x} - \beta_M(x,n)| = \sqrt{x} \leq \sqrt{\frac{n}{M}} \tag{80}$$

as stated. If $x \in [n/M, 1]$ we can check that the function

$$[n/M, 1] \ni x \mapsto \sqrt{x} - \sqrt{x - \frac{n}{M}} \in \mathbb{R} \tag{81}$$

is positive and monotonically decreasing (the latter can be easily seen by looking at the derivative). Hence the biggest value is achieved at the beginning of the interval (i.e. $x = n/M$) and we get again the statement of the lemma. □

Lemma 3.8. *For any pair of multiindices S, R and $n \in \mathbb{N}$ we have*

$$\lim_{M\to\infty} \|a^S(a^R_M(x) - x^{|R|/2}a^R)\psi_n\| = 0 \tag{82}$$

uniformly in x.

Proof. Note first that for $r \in \{-1, 1\}$

$$a^{(r)}_M\psi_n - \sqrt{x}a^{(r)}\psi_n = \left[\beta_M\left(x, n + \frac{r-1}{2}\right) - \sqrt{x}\right]\sqrt{n + \frac{r+1}{2}}\psi_{n+r} \tag{83}$$

The general case can be schematically written as

$$a^S(a^R_M(x) - x^{|R|/2}a^R)\psi_n \sim \prod_{j=1}^{d} \sqrt{n+p_j}\left(\beta_M(x, n+p'_j) - \sqrt{x}\right) a^S\psi_{n+q} \tag{84}$$

with $d = |R|$ and $p_i, p'_i, q \in \mathbb{N}$ can be easily written in terms of $w_j(R) \equiv \sum_{i=1}^{j} r_j$ with $R = (r_d, r_{d-1}, \ldots, r_1)$. Note, that the omitted factor of proportionality is M-independent. Now we have according to Lemma 3.7 for any $p \in \mathbb{N}$

$$\left|\sqrt{x} - \beta_M(x, n+p)\right| \leq \sqrt{\frac{n+p}{M}}. \tag{85}$$

Hence $\lim_{M\to\infty} \beta_M(x, n+p) = \sqrt{x}$ uniformly, which proves the statement. □

Now, we can prove convergence of $U_{M,t}(x)\psi_n$ for small times.

Lemma 3.9. *Let $x \in [0,1], n \in \mathbb{N}$ and $|t| < (32 \max|c_j|)^{-1}$. Then*

$$\lim_{M\to\infty} \|a^S U_{t,M}(x)\psi_n - a^S U_{t,x}\psi_n\| = 0 \tag{86}$$

uniformly in x.

Proof. Note first that the expressions in Equation (86) are well defined, since $U_{M,t}(x)\psi_n$ and $U_{x,t}\psi_n$ are Schwartz functions (this follows from Lemma 3.4) and therefore in the domain of a^S.

Now proceed with $q \equiv 32|t|\max|c_j| < 1$ which implies

$$\|a^S\frac{(itH)^m}{m!}\psi_n\| \leq 2^{3d-\frac{n}{2}}d!q^m\ . \tag{87}$$

The sum over $m \in \mathbb{N}$ is then convergent since the rhs. of (87) is also (since $q < 1$). Hence we can write $U_{x,t}\psi_n$ in terms of the exponential series

$$U_{x,t}\psi_n = \exp(-itxH)\psi_n = \sum_{n=0}^{\infty} \frac{(-itxH)^m \psi_n}{m!}. \tag{88}$$

Now note that $U_{x,t}\psi_n \in \mathcal{S}(\mathbb{R})$ and therefore $a^S U_{x,t}\psi_n$ is well defined. Hence we get

$$a^S U_{x,t}\psi_n = a^S \exp(-itxH)\psi_n = \sum_{n=0}^{\infty} \frac{a^S(-itxH)^m \psi_n}{m!}. \tag{89}$$

The last equality holds (although a^S is unbounded) because a^S is closable and convergence of the series is known from (87).

Now, consider

$$\begin{aligned}\|a^S(U_{M,t}(x) - U_{x,t})\psi_n\| = \|\exp(itH_M(x)) - \exp(itxH)]\psi_n\| \leq \\ \left\| \sum_{m>\mu} \frac{a^S(itH_M(x))^m}{m!}\psi_n \right\| + \left\| \sum_{m>\mu} \frac{a^S(itxH)^m}{m!}\psi_n \right\| + \\ \left\| \sum_{m\leq\mu} a^S \frac{(itH_M(x))^m - (itxH)^m}{m!}\psi_n \right\|\end{aligned} \tag{90}$$

Now, for all $\epsilon > 0$ there is $\mu_\epsilon \in \mathbb{N}$ (independent of M and x) such that both the first two terms are smaller than $\epsilon/3$, since they are bounded by the remainder of the geometric series c.f. (87) and the remark after it. For the third term we write

$$\left\| \sum_{m\leq\mu_\epsilon} a^S \frac{(itH_M(x))^m - (itxH)^m}{m!}\psi_n \right\| \leq \sum_{m\leq\mu_\epsilon} \|a^S\left(H_M(x)^m - x^m H^m\right)\psi_n\| \frac{|t|^m}{m!} \tag{91}$$

$$\leq \sum_{m\leq\mu_\epsilon} \sum_{R\in\{-1,1\}^{2m}} \left\| a^S \left(a_M^R(x) - x^{|R|/2} a^R \right) \psi_n \right\| \frac{|t|^m}{m!} \, . \tag{92}$$

The last expression contains finite number of terms, each of which converges (uniformly in x) to zero when $M \to \infty$ due to Lemma (3.8), hence for M large enough the last term in (90) is smaller than $\epsilon/3$ as well. □

This lemma leads to the following strong convergence result, which is a first step towards Theorem 2.10.

Proposition 3.10. *For any fixed* $\eta \in \mathcal{H}_\infty$

$$\lim_{M\to\infty} \|(U_{M,t}(x) - U_{x,t})\eta\| = 0 \tag{93}$$

holds uniformly in x.

Proof. Consider $\eta \in \mathcal{H}_\infty$ with $\|\eta\| = 1$ and its expansion in terms of ψ_n: $\eta = \sum_n \eta^{(n)}\psi_n$. For any $\epsilon > 0$ we can decompose $\eta = \eta_1 + \eta_2$ such that

$$\eta_1 = \sum_{n\leq n_\epsilon} \eta^{(n)}\psi_n, \quad \eta_2 = \sum_{n>n_\epsilon} \eta^{(n)}\psi_n, \quad \|\eta_2\| < \epsilon \tag{94}$$

Now we write

$$\|(U_{M,t}(x) - U_{x,t})\eta\| \leq \|(U_{M,t}(x) - U_{x,t})\eta_1\| + \|(U_{M,t}(x) - U_{x,t})\eta_2\| \tag{95}$$

$$\leq \sum_{n \leq n_\epsilon} \eta^{(n)} \|(U_{M,t}(x) - U_{x,t})\psi_n\| + 2\epsilon \tag{96}$$

According to Lemma (3.9) for each $n \leq n_\epsilon$ there is an $M_{n,\epsilon}$ such that for $M > M_{n,\epsilon}$ and all x we have $\|(U_{M,t}(x) - U_{x,t})\psi_n\| \leq \epsilon/n_\epsilon$ Since $\|\eta\| = 1$ we get

$$\|(U_{M,t}(x) - U_{x,t})\eta_1\| \leq n_\epsilon \frac{\epsilon}{n_\epsilon} = \epsilon \quad \forall M > M_\epsilon \equiv \max_{n \leq n_\epsilon} M_{n,\epsilon} \ . \tag{97}$$

Together with (96) this leads to $\lim_{M\to\infty} U_{M,t}(x)\eta = U_{x,t}\eta$. Since $\eta \in \mathcal{H}_\infty$ was arbitrary, the statement follows. □

Now we are aiming at convergence (in a sense we will make precise later) of $U_{M,t}(x)\rho U_{M,t}(x)^*$ to $U_{x,t}\rho U^*_{x,t}$. The next lemma is the first step.

Lemma 3.11. *$\forall n \in \mathbb{N}$ we have*

$$\lim_{M\to\infty} U_{M,t}(x) a^S U^*_{M,t}(x)\psi_n = U_{x,t} a^S U^*_{x,t}\psi_n \tag{98}$$

uniformly in x.

Proof. As in Lemma 3.9 note first that the expression in (98) is well defined since $U^*_{M,t}(x)\psi_n$ and $U_{x,t}\psi_n$ are in the domain of a^S.

From the same lemma we have in addition that $\Phi_{M,n} = a^S U^*_{M,t}(x)\psi_n$ converges uniformly to $\Phi_n = a^S U^*_{x,t}\psi_n$. Hence

$$\|U_{M,t}(x)\Phi_{M,n} - U_{x,t}\Phi_n\| \leq \|U_{M,t}(\Phi_{M,n} - \Phi_n)\| + \|(U_{M,t} - U_{x,t})\Phi_n\| < 2\epsilon \tag{99}$$

if $M > \max\{M_\epsilon, M'_\epsilon\}$ and for arbitrary x, where M_ϵ comes from the strong convergence of the second term proved in Prop. 3.10 whereas M'_ϵ from the above mentioned convergence. □

3.3. The key estimate

The purpose of this Section is to prove the following Lemma which will allow us to trace the convergence of sequences of unbounded operators $U^*_{M,t}(x) a^S U_{M,t}(x)$ back to convergence of bounded operators.

Lemma 3.12. *$\exists p, p' \in \mathbb{N} \cup \{\infty\}$ and $K \in \mathbb{R}_+$ such that there is a bounds*

$$\|(\mathbf{N} + p\mathbb{1})^{-\frac{p}{2}} U_{M,t}(x) a^S U^*_{M,t}(x)\| < K \tag{100}$$

and

$$\|U_{M,t}(x) a^S U^*_{M,t}(x) (\mathbf{N} + p'\mathbb{1})^{-\frac{p'}{2}}\| < K' \tag{101}$$

for sufficiently small t and $\forall M \in \mathbb{N}$ and for all $x \in [0,1]$.

Proof. Since ψ_n, $n \in \mathbb{N}$ is a basis, it is sufficient to show that

$$\|(\mathbf{N} + p\mathbb{1})^{-\frac{p}{2}} U_{M,t}(x) a^S U^*_{M,t}(x)\psi_n\| < K \tag{102}$$

and

$$\|U_{M,t}(x)a^S U^*_{M,t}(x)(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}}\psi_n\| < K \tag{103}$$

hold. Furthermore, it is sufficient to show this for $n > n_0$ since the overall bound will then be given by $\max\{\tilde{K}, K_n | n < n_0\}$ where we denoted $\tilde{K}$ the bound for $n > n_0$, and K_n is a bound for fixed n's in (102), which exists due to Lemma 3.11. For the proof of (102) and (103) $\forall n > n_0$ with a suitable $n_0 \in \mathbb{N}$ we proceed by introducing the spectral projections of the number operator $E_{(a,b)} = \chi_{[a,b]}(\mathbf{N})$ and proving some technical lemmas.

Lemma 3.13. *There is an $n_\epsilon \in \mathbb{N}$ and $t_\epsilon \in \mathbb{R}_+$ such that*

$$\|(\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi_{M,n}\| < \epsilon \textit{ with } \Phi_{M,n} = a^S U^*_{M,t}(x)\psi_n \tag{104}$$

$\forall M \in \mathbb{N},\ |t| < t_\epsilon,\ n > n_\epsilon.$

Proof. Since $a^S H^m_M(x)$ is a polynomial in $a_M(x), a^*_M(x)$ of order not greater than $d + 2m$ (with $d = |S|$), we have

$$(\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi^{(\mu)}_{M,n} = 0 \quad \text{if} \quad \frac{n}{2} \geq d + 2\mu \ , \tag{105}$$

where $\Phi^{(\mu)}_{M,n}$ denote the partial sums

$$\Phi^{(\mu)}_{M,n} = \sum_{m<\mu} \frac{a^S\big(itH_M(x)\big)^m}{m!}\psi_n. \tag{106}$$

Hence for the limit of the series the following formula holds:

$$(\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi_{M,n} = ((\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})(\Phi_{M,n} - \Phi^{((n-2d)/4)}_{M,n}) \ . \tag{107}$$

Let us estimate

$$\|(\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})(\Phi_{M,n} - \Phi^{((n-2d)/4)}_{M,n})\| \leq \|(\Phi_{M,n} - \Phi^{((n-2d)/4)}_{M,n})\| \tag{108}$$

$$\leq \sum_{m>\frac{n-2d}{4}} \frac{\|a^S H^m_M(x)\Psi_n\|}{m!}|t|^m \tag{109}$$

$$\leq 2^{3d+\frac{n}{2}} d! \sum_{m>\frac{n-2d}{4}} q^m \tag{110}$$

with $q = 32|t| \max|c_j|$. With sufficiently small t, c.f. Lemma 3.9, that is, $q < 1$ is assumed, we can sum the geometric series explicitly:

$$2^{3d+\frac{n}{2}} d! \frac{q^{\frac{n-2d}{4}+1}}{1-q} = \frac{2^{3d}d!}{1-q}\exp(\frac{2-d}{2}\log q)\exp[n(\log\sqrt{2} + \frac{1}{4}\log q)] \equiv K_{1,q}\, e^{-K_{2,q}n} \tag{111}$$

where the constants $K_{i,q}$ can be read off from the formula. Now shortening the time interval such that $q < 1/4$ we find make $K_{2,q} < 0$. Thus, we can conclude that there is an $n_{\epsilon,q}$ such that the statement of the Lemma is fulfilled. ☐

Lemma 3.14.

$$\|E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \leq \left(\frac{3n}{2} + 2d\right)^{\frac{d}{2}} \tag{112}$$

Proof. Let us expand $\Phi_{M,n}$ and $U_{M,n}(x)\psi_n$ in terms of Hermite functions

$$\Phi_{M,n} = \sum_k C_k \psi_k, \quad \phi = \sum_k c_k \psi_k \ . \tag{113}$$

Then $C_{k+w(S)} = \Lambda(k,S)c_k$ holds with $\Lambda(k,S)$ satisfying $(k-d)^{d/2} < \Lambda(k,S) < (k+d)^{d/2}$, where we recall $|S| = d$. Since the relation $-d \leq w(S) \leq d$ also holds, the condition $k + w(S) \in [n/2, 3n/2]$ implies $k \in [n/2-d, 3n/2+d]$. From these we get

$$\|E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\|^2 \leq \sum_j |c_{j-w(S)}|^2 (\frac{3n}{2} + 2d)^d \tag{114}$$

and since $\sum_k |c_k|^2 = 1$ due to $\|\psi_n\| = 1$ and the unitarity of $U_{M,t}(x)$ we finally get to the statement of the Lemma. □

Now let us return to the bound in Equation (102) and decompose the norm in the following way.

$$\begin{aligned}\|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} U_{M,t}\Phi_{M,n}\| \leq & \|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} E_{[\frac{n}{4},\frac{9n}{4}]} U_{M,t} E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \\ & + \|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} (\mathbb{1} - E_{[\frac{n}{4},\frac{9n}{4}]}) U_{M,t} E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \\ & + \|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} U_{M,t} (\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi_{M,n}\|\end{aligned} \tag{115}$$

The third term can be estimated as

$$\|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} U_{M,t} (\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi_{M,n}\| \leq \|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}}\| \|(\mathbb{1} - E_{[\frac{n}{2},\frac{3n}{2}]})\Phi_{M,n}\| \leq p^{-\frac{p}{2}}\epsilon \tag{116}$$

whenever $n > n_\epsilon$ due to Lemma 3.13. For the second term we write

$$\begin{aligned}&\|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} (\mathbb{1} - E_{[\frac{n}{4},\frac{9n}{4}]}) U_{M,t} E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \leq \\ &\qquad \|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}}\| \max_{k\in[\frac{n}{2},\frac{3n}{2}]} \|(\mathbb{1} - E_{[\frac{n}{4},\frac{9n}{4}]}) U_{M,t}\psi_k\| \|E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \leq \\ &\qquad\qquad p^{-\frac{p}{2}} K_{1,q} e^{-K_{2,q}\frac{n}{2}} \left(\frac{3n}{2} + 2d\right)^{\frac{d}{2}} \ ,\end{aligned} \tag{117}$$

where we used the Lemma 3.14 and the technique of the proof of Lemma 3.13 for the middle and right norms in the second line to arrive at the third. Note, that for $k \in [n/2, 3n/2]$ we have that $[k/2, 3k/2] \supset [n/4, 9n/4]$, which justifies using the same constants $K_{i,q}$ as in Lemma 3.13. It is now clear, that increasing n makes this bound arbitrary small. Finally, note that $(\mathbf{N}+p\mathbb{1})^{-p/2}$ commutes with $E_{[n/4, 9n/4]}$, and since the latter is a projection we can write for the first term

$$\begin{aligned}&\|(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} E_{[\frac{n}{4},\frac{9n}{4}]} U_{M,t} E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \leq \\ &\qquad \|E_{[\frac{n}{4},\frac{9n}{4}]}(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} E_{[\frac{n}{4},\frac{9n}{4}]}\| \|U_{M,t}\| \|E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \ ,\end{aligned} \tag{118}$$

Now, for the norms we have

$$\|E_{[\frac{n}{4},\frac{9n}{4}]}(\mathbf{N}+p\mathbb{1})^{-\frac{p}{2}} E_{[\frac{n}{4},\frac{9n}{4}]}\| \leq 2^p (n+4p)^{-\frac{p}{2}}, \quad \|E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}\| \leq \left(\frac{3n}{2} + 2d\right)^{\frac{d}{2}} \ , \tag{119}$$

the multiplication of which, after substitution $p = d$, gives $6^d r^{d/2} < 6^d$ since $r \equiv (n+4d/3)/(n+4d) < 1$. Hence we have found $\tilde{K} = 6^d + 2\epsilon$, which leads to (102) and therefore (100). To

get (103) and (101) we can proceed in exactly the same way. The only difference is that the operator $(\mathbf{N}+p'\mathbb{1})^{-\frac{p'}{2}}$ is acting directly on ψ_n and not at $U_{M,t}E_{[\frac{n}{2},\frac{3n}{2}]}\Phi_{M,n}$. Therefore we only have to replace the first norm in (119) by $(n+p')^{p'/2}$ to get the desired result and the proof is complete. □

Now we are ready to prove a convergence statement for the time evolution $U_{M,t}(x)\rho U^*_{M,t}(x)$ which brings a step closer to our goal.

Proposition 3.15. *For any Schwartz operator $\rho \in \mathcal{S}(\mathcal{H}_\infty)$ we have*

$$\lim_{M\to\infty} \mathrm{Tr}[a^S U_{M,t}(x)\rho U^*_{M,t}(x)] = \mathrm{Tr}[a^S U_{x,t}\rho U^*_{x,t}] \tag{120}$$

uniformly in x.

Proof. According to Lemma 3.4 Equation (120) is equivalent to

$$\lim_{M\to\infty} \mathrm{Tr}[U^*_{M,t}a^S U_{M,t}(x)\rho] = \mathrm{Tr}[U^*_{x,t}a^S U_{x,t}\rho] \tag{121}$$

To prove this let us introduce the notations

$$X_M(x) \equiv (\mathbf{N}+p\mathbb{1})^{-\frac{d}{2}}U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\,, \quad X(x) \equiv (\mathbf{N}+p\mathbb{1})^{-\frac{d}{2}}U^*_{x,t}\, a^S\, U_{x,t}\ , \tag{122}$$

and show first strong convergence of the defined operators.

Lemma 3.16. *For any $\eta \in \mathcal{H}_\infty$ we have*

$$\lim_{M\to\infty} \|(X_M(x)-X(x))\eta\| = 0 \tag{123}$$

uniformly in x.

Proof. We use the same strategy as for Proposition 3.10 and decompose $\eta \in \mathcal{H}_\infty$ as

$$\eta_1 = \sum_{n\le n_\epsilon} \eta^{(n)}\psi_n, \quad \eta_2 = \sum_{n> n_\epsilon} \eta^{(n)}\psi_n, \quad \|\eta_2\| < \epsilon\,, \tag{124}$$

cf. formula (94). Then

$$\begin{aligned} \|(X_M(x)-X(x))\eta\| &\le \|(X_M(x)-X(x))\eta_1\| + \|(X_M(x)-X(x))\eta_2\| && (125)\\ &\le \sum_{n\le n_\epsilon} |\eta^{(n)}| \|(X_M(x)-X(x))\psi_n\| + 2K\epsilon && (126)\\ &\le \Big(\sum_{n\le n_\epsilon} |\eta^{(n)}|\Big)\, \|(\mathbf{N}+d\mathbb{1})^{-\frac{d}{2}}\| \frac{\epsilon}{n_\epsilon} + 2K\epsilon\,, && (127) \end{aligned}$$

where the second inequality is the consequence of Lemma 3.12, whereas for the third one assumes that $M > \max_{n<n_\epsilon}(M_\epsilon)$ in the statement of Lemma 3.11. Since $\big(\sum_{n\le n_\epsilon}|\eta^{(n)}|\big) \le n_\epsilon$ and $(\mathbf{N}+d\mathbb{1})^{-\frac{d}{2}}$ is bounded, the convergence is proved. □

The sequence X_M converges strongly and is norm bounded. This implies for any trace class operator convergence of traces $\mathrm{Tr}(X_M(x)\rho)$ with $\lim_{M\to\infty}\mathrm{Tr}(X_M(x)\rho) = \mathrm{Tr}(X(x)\rho)$. Hence we get

$$\lim_{M\to\infty} \big|\mathrm{Tr}\big(X_M(x)\rho\big) - \mathrm{Tr}\big(X(x)\rho\big)\big| = 0 \tag{128}$$

uniformly in x. Therefore we get with a *Schwartz operator* ρ

$$\lim_{M\to\infty} \mathrm{Tr}(U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\,\rho) = \lim_{M\to\infty} \mathrm{Tr}\left((\mathbf{N}+p\mathbb{1})^{\frac{d}{2}}(\mathbf{N}+p\mathbb{1})^{-\frac{d}{2}}U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\,\rho\right) \quad (129)$$

$$= \lim_{M\to\infty} \mathrm{Tr}\left(X_M \rho (\mathbf{N}+p\mathbb{1})^{\frac{d}{2}}\right) \quad (130)$$

$$= \lim_{M\to\infty} \mathrm{Tr}\, X_M \tilde{\rho} = \mathrm{Tr}\, X \tilde{\rho} = \mathrm{Tr}(U^*_{x,t}\, a^S\, U_{x,t}\rho) \quad (131)$$

where $\tilde{\rho}$ is again a Schwartz operator (and therefore trace class) according to Proposition 3.1. □

3.4. Proof of the main theorem

Based on the result derived so far we can now start to prove the main theorem. This means in particular that we have to take the x-dependence stronger into account. We start with a short lemma which will be the main tool throughout this Subsection.

Lemma 3.17. *For each $M \in \mathbb{N} \cup \{\infty\}$, $x \in [0,1]$, $t \in \mathbb{R}$ with $|t|$ sufficiently small and each multiindex $S \in \{-1,1\}^d$ (including the trivial case $|S| = d = 0$) let us define*

$$\omega^S_{M,t,x}(\rho) = \mathrm{Tr}\big(U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\,\rho\big) \quad (132)$$

and consider $\Sigma \subset \mathcal{S}(\mathcal{H})$ bounded, i.e.,

$$\|\rho\,(\mathbf{N}+p\mathbb{1})^{p/2}\|_1 < K_p \;\forall \rho \in \Sigma. \quad (133)$$

Then we have

$$|\omega^S_{M,t,x}(\rho)| \le K K_p \quad (134)$$

with the constant K from Lemma 3.12 and $p \in \mathbb{N}$ sufficiently big (such that the statement from Lemma 3.12 holds). Moreover $\omega^S_{M,t,x}(\rho)$ is continuous as a function of x.

Proof. Since ρ is a Schwartz operator we can rewrite $\omega^S_{M,t,x}$ as

$$\omega^S_{M,t,x}(\rho) = \mathrm{Tr}\big((\mathbf{N}+p\mathbb{1})^{-p/2}U_{M,t}(x)^* a^S U_{M,t}(x)\rho\,(\mathbf{N}+p\mathbb{1})^{p/2}\big) \quad (135)$$

hence

$$|\omega^S_{M,t,x}(\rho)| \le \|(\mathbf{N}+p\mathbb{1})^{-p/2}U_{M,t}(x)^* a^S U_{M,t}(x)\|\|\rho\,(\mathbf{N}+p\mathbb{1})^{p/2}\|_1 \le K\cdot K_p \quad (136)$$

The first term is bounded by K by Lemma 3.12, the second term is bounded by K_p by assumption (where p has to be chosen large enough such that Lemma 3.12 is true). This shows boundedness.

To show continuity in x consider first the case $M = \infty$. For a quadratic Hamiltonian $\exp(-itH)a^S\exp(itH)$ is a linear differential operator in x with coefficients smoothly depending on t. Hence the continuity of $\omega^S_{\infty,x,t}(\rho)$ in x follows from that of $\exp(-ixtH)$ for each t fixed.

If M is finite the situation is effectively finite dimensional, since $U_{M,t}(x)$ acts as the identity on all ψ_n with $n > Mx$; the monomial a^S can map ψ_n to $\psi_{n+|S|}$; hence, we have to look at the Hilbert space spanned by ψ_n, $n = 0,1,2,\ldots,xM+|S|$. Now, continuity of $\omega^S_{M,x,t}(\rho)$ follows from the continuity of the map $x \mapsto H_M(x)$ in norm for all fixed λ, M, since this continuity implies norm continuity of $U_{M,t}(x)$ in x $\forall M,t$. □

Lemma 3.18. *Consider a sequence of continuous maps: $R_{M,\lambda} : [0,1] \to \mathcal{S}(\mathcal{H})$, $M \in \mathbb{N}$ such that*

1. *$\{R_{M,\lambda}(x) | x \in [0,1], M \in \mathbb{N}\}$ is a bounded subset of $\mathcal{S}(\mathcal{H})$*

2. $R_{M,\lambda}(x)$ converges in $\mathcal{S}(\mathcal{H})$ and uniformly on a neighbourhood $I \subset [0,1]$ of λ to a continuous function $R_\infty : [0,1] \to \mathcal{S}(\mathcal{H}_\infty)$.

Then the sequence

$$\mathrm{Tr}\left(U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\, R_{M,\lambda}(x) - U^*_{x,t}\, a^S\, U_{x,t} R_\infty(x)\right) \tag{137}$$

is uniformly bounded on $x \in [0,1]$ and converges uniformly to 0 on $x \in I$.

Proof. Convergence of the $R_{M,\lambda}(x)$ implies that for each $p \in \mathbb{N}$ and each ϵ we can find an $M_{p,\epsilon}$ such that

$$\|(R_{M,\lambda}(x) - R_\infty(x))(\mathbf{N} + p\mathbb{1})^{p/2}\| < \frac{\epsilon}{K} \quad \forall M > M_{p,\epsilon} \tag{138}$$

holds. With Lemma 3.17 this implies

$$|\omega^S_{M,t,x}(R_{M,\lambda}(x) - R_\infty(x))| \leq K\frac{\epsilon}{K} = \epsilon,\ \forall x \in I, \forall M > M_\epsilon \tag{139}$$

Hence, we write

$$\begin{aligned}\left|\mathrm{Tr}\big(U^*_{M,t}(x)\, a^S\, U_{M,t}(x)\, R_{M,\lambda}(x) - U^*_{x,t}\, a^S\, U_{x,t} R_\infty(x)\big)\right| \leq \\ \left|\omega^S_{M,t,x}\big(R_{M,\lambda}(x) - R_\infty(x)\big)\right| + \left|\omega^S_{M,t,x}\big(R_\infty(x)\big) - \omega^S_{\infty,t,x}(R_\infty(x))\right|,\end{aligned} \tag{140}$$

where the first term was just shown to be $< \epsilon$ and the second term converges to zero by Proposition 3.15. Note, that both statement are independent of $x \in I$.

Boundedness follows similarly, if use condition 1 and the fact that the map $R_\infty(x) \in \mathcal{S}(\mathcal{H}_\infty)$ is bounded as well (due to continuity and compactness of $[0,1]$). □

Lemma 3.19. *Consider $M \in \mathbb{N} \cup \{\infty\}$, $x \in [0,1]$, $t \in \mathbb{R}$ with $t/$ sufficiently small, a multiindex $S \in \{-1,1\}^d$ and a continuous map $R : [0,1] \to \mathcal{S}(\mathcal{H}_\infty)$. Then the functions*

$$[0,1] \ni x \mapsto \omega^S_{M,x,t}\big(R(x)\big) \in \mathbb{C} \tag{141}$$

are continuous.

Proof. Consider a sequence $x_j \in [0,1]$, $j \in \mathbb{N}$ converging to $x \in [0,1]$. Continuity of R implies that for each $\epsilon > 0$ and each $r \in \mathbb{N}$ there is an $j_{r,\epsilon}$ such that

$$\|(\mathbf{N} + r\mathbb{1})^r \rho\| < \epsilon \tag{142}$$

holds for all ρ in the set

$$\Sigma_{r,\epsilon} = \{R(x_j) - R(x) \,|\, j > j_{\epsilon,r}\}. \tag{143}$$

Hence $j > j_{r,\epsilon}$ implies

$$\begin{aligned}\big||\omega^S_{M,x_j,t}\big(R(x_j)\big) - \omega^S_{M,x,t}\big(R(x)\big)\big| \leq \\ \big||\omega^S_{M,x_j,t}\big(R(x_j) - R(x)\big)\big| + \big|\omega^S_{M,x_j,t}\big(R(x)\big) - \omega^S_{M,x,t}\big(R(x)\big)\big|.\end{aligned} \tag{144}$$

The first term on the right hand side is bounded by $K\epsilon$ due to (142) and Lemma 3.17. The second term can be made arbitrarily small due to continuity of $\omega^S_{M,x,t}(\rho)$ in x; cf. again Lemma 3.17. □

Lemma 3.20. *The sequence*

$$M \mapsto (a_M(x)^S - x^{\frac{|S|}{2}} a^S)(\mathbf{N} + 2|S|\mathbb{1})^{-|S|} \ . \tag{145}$$

converges to zero in norm uniformly in $x \in [0,1]$ *any closed subinterval of* $x \in I \subset [0,1]$ *not containing* 0.

Proof. The statement is proved if we can show that for each $\epsilon > 0$ we can find an M_ϵ which is *independent of* n *and* x such that

$$\|a_M^S(x) - x^{\frac{|S|}{2}} a^S)(\mathbf{N} + 2|S|\mathbb{1})^{-|S|}\psi_n\| < \epsilon\epsilon \tag{146}$$

holds for all $M > M_\epsilon$. Using the definition of $a_M(x)$ in Equations (38) and (39) and properties of the standard creation and annihilation operators it is easy to see that the elements of the sequence in (146) are products of $|S|$ terms of the form

$$\big(\beta_{M,\lambda}(x, n+r) - \sqrt{x}\big)\sqrt{n+r'}\,(n+2|S|)^{-1} \tag{147}$$

which differ only in the parameter r, r'; we have used a similar argument already in the proof of Lemma 3.8; cf. Equation (84). The only difference are $|S|$ terms $(n+2|S|)^{-1}$ we have distributed among the S factors. Using the bound in lemma 3.7 we get

$$\frac{\beta_{M,\lambda}(x, n+r) - \sqrt{x}}{\sqrt{n+2|S|}} \frac{\sqrt{n+r'}}{\sqrt{n+2|S|}} \leq \frac{1}{\sqrt{M}} \sqrt{\frac{n+r}{n+2|S|} \frac{n+r'}{n+2|S|}} \tag{148}$$

The right hand side can be bound in an n-independent way to

$$\sqrt{\frac{n+r}{n+2|S|} \frac{n+r'}{n+2|S|}} \leq \sqrt{\max(1, r/2S)\max(1, r'/2S)}. \tag{149}$$

This leads to the estimate in (146) which concludes the proof. □

Lemma 3.21. *For each* $p \in \mathbb{N}$, $t \in \mathbb{R}$ *sufficiently small and a uniformly bounded sequence* $R_M : [0,1] \to \mathcal{S}(\mathcal{H}_\infty)$ *the sequence*

$$\|(\mathbf{N} + 2p\mathbb{1})^p U_{M,t}(x) R_M(x) U_{M,t}^*(x)\|_1 \quad M \in \mathbb{N} \tag{150}$$

is uniformly bounded in x.

Proof. Since $(\mathbf{N} + 2p\mathbb{1})^p$ is a polynomial in a and a^* it is sufficient to look at

$$\|a^S U_{M,t}(x) R_M(x) U_{M,t}^*(x)\|_1. \tag{151}$$

In addition we can rewrite this expression as

$$\begin{aligned} \|a^S U_{M,t}(x) R_M(x) U_{M,t}^*(x)\|_1 &= \|U_{M,t}^*(x) a^S U_{M,t}(x) R_M(x)\|_1 & (152)\\ &= \|U_{M,t}^*(x) a^S U_{M,t}(x)(\mathbf{N} + p'\mathbb{1})^{-p'/2}(\mathbf{N} + p'\mathbb{1})^{p'/2} R_M(x)\|_1 & (153)\\ &\leq \|U_{M,t}^*(x) a^S U_{M,t}(x)(\mathbf{N} + p'\mathbb{1})^{-p'/2}\|\|(\mathbf{N} + p'\mathbb{1})^{p'/2} R_M(x)\|_1 \leq K' K_1 & (154) \end{aligned}$$

The first factor is bounded according to Lemma 3.12 the second because R_M is uniformly bounded by assumption and $\rho \mapsto a^S\rho$ is a continuous map on $\mathcal{S}(\mathcal{H}_\infty)$; cf. Proposition 3.1. □

Lemma 3.22. *For each $M \in \mathbb{N}$, $t \in \mathbb{R}$ sufficiently small and $R \in \{-1,1\}^d$ the function*

$$[0,1] \ni x \mapsto \left|\mathrm{Tr}\big(U_{M,t}(x) R_M(x) U^*_{M,t}(x)(a^R_M(x) - a_\infty(x)^R)\big)\right| \in \mathbb{C} \tag{155}$$

is continuous.

Proof. We can write this expression as a linear combination of therms $\omega^S_{M,x,t}\big(R_M(x)\big)$ with coefficients depending continuously on x. Hence the statement follows from Lemma 3.19. □

Now we are ready to finish the proof of Theorem 2.10. To this end let us go back to Equation (55) which was

$$\begin{aligned}\left|\mathrm{Tr}(U_{M,t}\rho_M U^*_{M,t} a^R_M) - \mathrm{Tr}(U_{\lambda,t}\rho_\infty U^*_{\lambda,t} a^R)\right| < \\ \int_0^1 \left|\mathrm{Tr}\big(U_{M,t}(x) R_M(x) U^*_{M,t}(x)(a^R_M(x) - a_\infty(x)^R)\big)\right| \mu_M(dx) + \\ \int_0^1 \left|\mathrm{Tr}\big((U_{\lambda,t} R_\infty(x) U^*_{\lambda,t} - U_{M,t}(x) R_\infty(x) U^*_{M,t}(x)) a_\infty(x)^R\big)\right| \mu_M(dx)\end{aligned} \tag{156}$$

where $a_\infty(x) = x^{|R|/2} a$ according to Equation (51).

The integrand of the second integral is – according to Lemmas 3.18 and 3.19 – continuous and uniformly bounded in $x \in [0,1]$, and it converges uniformly on $x \in I$ to 0. In addition we know that the measures μ_M converge weakly to the point measure at λ. Hence the second integral vanishes in the limit $M \to 0$.

The integrand of the first integral is continuous as well (Lemma 3.22) and we can rewrite it according to

$$\begin{aligned}&\left|\mathrm{Tr}\big((a^R_M(x) - x^{R/2}a^R)(\mathbf{N} + 2|S|\mathbb{1})^{-|S|}(\mathbf{N} + 2|S|\mathbb{1})^{|S|} U_{M,t}(x) R_M(x) U^*_{M,t}(x)\big)\right| \\ &\quad \leq \|(a^R_M(x) - x^{R/2}a^R)(\mathbf{N} + 2|S|\mathbb{1})^{-|S|}\| \|(\mathbf{N} + 2|S|\mathbb{1})^{|S|} U_{M,t}(x) R_M(x) U^*_{M,t}(x)\|_1.\end{aligned} \tag{157}$$

The first factor on the right hand side converges to 0 uniformly in x (Lemma 3.20) while the second is uniformly bounded (Lemma 3.21). Hence in the limit $M \to \infty$ this integral vanishes as well, which concludes the proof.

4. Outlook

We have shown that a one degree of freedom, continuous variable quantum system can be simulated by mean field fluctuations of an ensemble of qubits. Our results substantially exceed existing schemes. This includes in particular the range of states we can simulate and the treatment of the dynamics. Nevertheless, there are a number of open questions which needs to be treated in forthcoming papers.

- **Dynamics.** Our results concerning the dynamics have two essential restrictions: The statements are restricted to quadratic Hamiltonians and to small times. The latter is probably only a restriction of the methods in the proof and can be removed with a more careful analysis of the limiting behavior of the unitaries $U_{M,t}(x)$. One possibility is to show that the time evolved density operators $U_{M,t}(x) R_M(x) U^*_{M,t}(x)$ converge in the same as the §$R_M(x)$ do, because in this case we can stack arbitrarily many finite time intervals together.

  The discussion of more general Hamiltonians is most likely a more difficult problem. In Section 3.2 we have made heavy use of the fact that Hermite functions are analytic vectors

for *all* quadratic Hamiltonians, and this argument has to be replaced somehow. A minimal requirement is in any case that the dynamics leaves the space of Schwartz functions invariant. The questions are: Which Hamiltonians have this property and is it sufficient to prove an analog of Theorem 2.10?

- **The reference state.** The reference state θ plays a very crucial role in the theory, which was overlooked in previous publications. It provides the parameter λ which plays in some respect the role of an effective $\hbar$, although this is not universally true, as we have seen during the discussion of Theorem 2.9. In particular the limit $\lambda \to 0$ seems to be boring, because in our scheme the observables Q_∞ and P_∞ just scale down with $\lambda^{-1/2}$ and therefore all expectation values vanishes if λ approaches 0. However, this is not necessarily the end of the story. Instead of using sequences of states converging according to Definition 2.8 we can use more divergent ones such that expectation values do not vanish in the limit $M \to \infty$. Alternatively we can rescale $Q_M(x)$ and $P_M(x)$ by $x^{-1/2}$ and $x^{1/2}$ respectively.

  Another important aspect arises in connection dynamics. The time evolution we have looked at in this paper leave the references state invariant (at least asymptotically in the limit $M \to \infty$). It would be interesting to break this property and consider situations where the reference state is time-dependent.

- **Classical observables.** Throughout our analysis we have basically ignored the fluctuation operator $F_M(\sigma_3)$. A short calculation along the lines of Section 2.1 shows that $F_M(\sigma_3)$ commutes in the limit $M \to \infty$ with Q_M and P_M, hence it describes a classical observables. The fact that $\mathrm{Tr}(\sigma_3\theta) = 0$ holds indicates that it is related to λ.

- **Higher dimensions.** To extend of our schemes to systems with arbitrary, finite dimensional one-particle spaces is a very natural task, and the generalization of the discussion in Section 2.1 is straightforward. The analysis of permutation invariant states along the lines of Section 2.3 is not. This concerns in particular the embedding of irreducible representation spaces of unitary groups into Hilbert spaces $\mathrm{L}^2(\mathbb{R}^d)$.

- **Gaussian states.** A lot of theory is available for Gaussian states and it is not completely clear how it is related to our approach. In [6] it is shown that a translation invariant, exponentially clustering state ω of an infinite spin chain leads to Gaussian fluctuations if restrictions of ω to finite parts of the chain are used as the sequence ρ_M. In this case it clear already from the analysis of correlation functions (as in Section 2.1) that the operators Q_∞ and P_∞ can be realized as canonical position and momentum in the Schrödinger representation (modulo factors of the form $\lambda^{\pm 1/2}$). Also the density operator ρ_∞ is easily calculated since only one and two point correlation functions are needed. It is, however, not clear whether these sequences converge as described in Section 2.3.

  Another interesting aspect concerns the relation between Gaussianity and correlations. The analysis in [6] uses in a crucial way that the ρ_M arises from restrictions of a thermodynamic limit state. Our scheme, on the other hand, does not need this assumption and it would be interesting up to which degree absence of long range correlations (in a to be specified limiting sense) still implies Gaussianity. Even more interesting (and more ambitious as well) is question whether we can distinguish classical correlations and entanglement by properties of the limit state ρ_∞.

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A. Fluctuation operators and representations of tensor algebras

To prove Proposition 2.2 let us consider the tensor algebra $\mathcal{A}$ over the vector space $V = \mathbb{C}^2$:

$$\mathcal{A} = \bigoplus_{M \in \mathbb{N}} V^{\otimes M}, \quad V^{\otimes 0} = \mathbb{C}. \tag{158}$$

Together with the tensor product and the involution

$$(x^{(1)} \otimes \cdots \otimes x^{(k)})^* = \bar{x}^{(k)} \otimes \cdots \otimes \bar{x}^{(1)}, \tag{159}$$

(where $\bar{x}$ denotes complex conjugation in the canonical basis of $V = \mathbb{C}^2$) the space $\mathcal{A}$ becomes a *-algebra. The vector space V is naturally embedded in $\mathcal{A}$ and its canonical basis

$$q = \frac{(1,0)}{\sqrt{2}} \in V, \quad p = \frac{(0,1)}{\sqrt{2}} \in V \tag{160}$$

is a complete set of generators.

The fluctuation operators F_M give now rise to a *-representation of $\mathcal{A}$ by

$$\Phi_M(q) = Q_M, \quad \Phi_M(p) = P_M \tag{161}$$

where $Q_M = 2^{-1/2} F_M(\sigma_1)$, $P_M = 2^{-1/2} F_M(\sigma_2)$ are the operators defined in Equation (5). Any density operator $\rho_M \in \mathcal{B}(\mathcal{H}^{\otimes M})$ leads to a state

$$W_M(X) = \mathrm{Tr}(\Phi_M(X)\rho_M), \quad X \in \mathcal{A} \tag{162}$$

of the algebra and if the sequence ρ_M, $M \in \mathbb{N}$ has $\sqrt{M}$ fluctuations the limit

$$W(X) \lim_{M \to \infty} W_M(X) = \mathrm{Tr}(\Phi_M(X)\rho_M) \tag{163}$$

exists for all X and defines again a state W of $\mathcal{A}$.

The state W defines the GNS representation[5] $(\mathcal{H}_W, D_W, \pi_w, \Omega_W)$ of $\mathcal{A}$ consisting of a Hilbert space H_W, a dense subspace D_W, a *-morphism π_W into the algebra of (unbounded) operators $D_W \to D_W$ and a cyclic vector Ω_W satisfying

$$\langle \Omega_W, \pi_w(X)\Omega_W \rangle = W(X). \tag{164}$$

Without loss of generality we will assume in the following that D_W coincides with the subspace generated by vectors $\pi_W(X)\Omega_W \in D_W$. Applying π_W to the generators q, p from Equation (160) we get two (unbounded) operators

$$Q_\infty = \pi_W(q), \quad P_\infty = \pi_W(p) \tag{165}$$

[5] Usually the GNS representation is only considered for C*-algebras, but it can easily be generalized to more general *-algebras. We only have to allow representations in terms of unbounded operators. Our case is basically an instance of the celebrated reconstruction theorem of Wightman quantum field theory [12].

with domain D_W. For any polynomial $f(q,p)$ they satisfy

$$\lim_{M\to\infty} \mathrm{Tr}\big(f(Q_M,P_M)\rho_M\big) = \mathrm{Tr}\big(f(Q_\infty,P_\infty)\rho_\infty\big) \quad \text{with} \quad \rho_\infty = |\Omega_W\rangle\langle\Omega_W|. \tag{166}$$

This establishes all statements in Proposition 2.2 except the last commutation relations (item 4).

To show the latter, let us consider two elements $X_1, X_2 \in \mathcal{A}$ and the expression (cf. Equation (7))

$$\begin{aligned}\mathrm{Tr}\big(\Phi_M(X_1)[Q_M,P_M]\Phi_M(X_2)\big) - i\lambda\,\mathrm{Tr}\big(\Phi_M(X_1)\Phi_M(X_2)\rho_M\big)&\\ = \frac{i}{2\sqrt{M}}\,\mathrm{Tr}\big(\Phi_M(X_1)F_M(\sigma_3)\Phi_M(X_2)\big).&\end{aligned} \tag{167}$$

If the sequence ρ_M has $\sqrt{M}$ fluctuations we have

$$\lim_{M\to\infty} \mathrm{Tr}\big(\Phi_M(X_1)F_M(\sigma_3)\Phi_M(X_2)\big) = C < \infty, \tag{168}$$

hence

$$\begin{aligned}&W\big(X_1(qp-pq-i\lambda\mathbb{1})X_2\big)\\ &\quad= \lim_{M\to\infty}\big[\mathrm{Tr}\big(\Phi_M(X_1)[Q_M,P_M]\Phi_M(X_2)\big) - i\lambda\,\mathrm{Tr}\big(\Phi_M(X_1)\Phi_M(X_2)\rho_M\big)\big] = 0.\end{aligned} \tag{169}$$

In other words, the limiting state W vanishes on the ideal generated by $qp-pq-i\lambda\mathbb{1}$ which leads to

$$[Q_\infty, P_\infty]\phi = i\lambda\phi \quad \forall\phi\in D_W \tag{170}$$

as stated.

References

[1] B. Julsgaard, J. Sherson, J. Cirac, J. Fiurasek and E. S. Polzik, *Experimental demonstration of quantum memory for light*, Nature **432**, 482 - 486, (2004)

[2] T. Holstein and H. Primakoff, *Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet*, Phys. Rev. **58**(12), 1098, (1940)

[3] K. Hammerer, A. S. Sorensen, E. S. Polzik, *Quantum interface between light and atomic ensembles*, Rev. Mod. Phys. **82**, 1041, (2010)

[4] H. Narnhofer and W. Thirring, *On entanglement properties of Mesoscopic Systems*, Phys. Rev., A. **66**, 052304, (2002); H. Narnhofer, *The time evolution of fluctuation algebra in mean field theories*, Found. Phys. **17**3, 235-253 (2004)

[5] G. A. Raggio and R. F. Werner, *Quantum statistical mechanics of general mean field systems*, Helv. Phys. Acta, **62**8, 980-1003, (1989); D. Goderis, A. Verbeure and P. Vets, *Non-commutative central limits*, Prob. Theor. Rel. Fields **82**, 527-544, (1989); D. Goderis and P. Vets, *Central limit theorem for mixing quantum systems and the CCR-algebra of fluctuations*, Comm. Math. Phys. **122**, 249-265, (1989)

[6] T. Matsui, *On the Algebra of Fluctuation in Quantum Spin Chains*, Ann. Henri Poincaré **4**1, 63-83, (March 2003)

[7] M. Keyl, J. Kiukas, R. Matjeschk and R. F. Werner, *Schwarz operators*, in preparation

[8] M. Keyl and R. F. Werner, *The rate of optimal purification procedures*, Ann. Henri Poincaré **2**, 1-26 (2001)

[9] G. Tóth and M. W. Mitchell, *Generation of macroscopic singlet states in atomic ensembles*, New J. Phys. **12**, 053007, (2010)

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, New York Academic (1975)

[11] G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press (1989)

[12] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Princeton Landmarks in Mathematics and Physics, Princeton University Press (2000)